Exam Numerieke Wiskunde 1
January 27, 2011

Use of “grafische rekenmachine“ is allowed.
All answers need to be motivated
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 6 points can be scored with this exam.

1. In the table below the results of a computation with a fixed point method $x_{n+1} = g(x_n)$ are given

n	x_n
0	1
1	1.53846
2	1.29502
3	1.40183
4	1.35421
5	1.37530
6	1.36593
7	1.37009

   (a) $\boxed{2}$ Give an estimate of $g'(p)$ at x_7 where p is the fixed point.

   (b) $\boxed{3}$ Use the estimate of $g'(p)$ in (a) to give the Aitken error estimate for x_7.

   (c) $\boxed{3}$ How can the error estimate of part (b) be used to improve x_7?

2. Suppose $f_1 = 4x_1^2 + x_2^2 - 4, \quad f_2 = x_1 + x_2 - \sin(x_1 - x_2)$. We want to solve $f_1(x_1, x_2) = 0, \ f_2(x_1, x_2) = 0$, which is written in vector form as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. We want to solve it with a fixed point method $\mathbf{x_{n+1}} = \mathbf{g}(\mathbf{x_n})$ where $\mathbf{g}(\mathbf{x}) = \mathbf{x} + A\mathbf{f}(\mathbf{x})$.

   (a) $\boxed{3}$ Give the Jacobian matrix of $\mathbf{f}$.

   (b) $\boxed{1}$ Why is $(1, 0)$ a reasonable guess of the zero?

   (c) $\boxed{4}$ Give the equation from which the best matrix A, based on the guess in the previous part, must be solved.

   (d) $\boxed{3}$ The Jacobian of $\mathbf{g}$ at the fixed point using the A of the previous part is

$$\begin{bmatrix} 0.0014 & 0.026375 \\ -0.0590842 & 0.0507948 \end{bmatrix}$$

   Why is this matrix relevant for the study of the convergence of the fixed point method? Will the method converge in the neighborhood of the fixed point and why?

3. Given the disjunct pairs $(x_i, f(x_i))$ for $i = 0, 1, 2$.

   (a) $\boxed{3}$ Give the Lagrange interpolating polynomial through these data points.

   (b) $\boxed{3}$ Give the interpolating polynomial through these data points expressed in Newton divided differences.

   (c) $\boxed{2}$ What is the advantage of the interpolating polynomial based on divided differences with respect to the Lagrange interpolation polynomial?

Continues on other side!

4. (a) [2] Using a sketch, make clear that both the midpoint rule and the trapezium rule are exact integration rules for linear functions.

   (b) [4] Suppose we have a numerical method which approximates I by $I(h)$, where h is the mesh size and $I = I(h) + ch^3 + O(h^5)$ for some nonzero c. Derive a combination of $I(h)$ and $I(2h)$ that approximates I and which has error $O(h^5)$.

5. Consider the problem

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ with } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   (a) [3] Give the difference equations that result from applying the trapezium method.

   (b) [4] Derive and draw the regions of absolute stability for the forward Euler (or explicit Euler), the backward Euler (or implicit Euler) and the trapezium method (or Crank Nicholson method).

   (c) [2] Which of the methods in the previous part allow a stable integration of the system above and why?

6. Consider the system $Ax = b$ with

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (a) [2] Give the iteration that results from applying the Jacobi method to this system.

   (b) [2] Does this method converge and why?

   (c) [2] Give the iteration that results from applying the Gauss Seidel method.

7. Consider on [0, 1] for $u(x,t)$ the diffusion equation $\partial u/\partial t = \kappa \partial^2 u/\partial x^2$, with $\kappa = 1$, and initial condition $u(x,0) = \sin(\pi x)$ and boundary conditions $u(0,t) = 0$ and $u(1,t) = \sin^2(t)$. Let the grid in x direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

   (a) [3] Show that $u_{xx}(x_i,t) = \frac{u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)}{\Delta x^2} + O(\Delta x^2)$

   (b) [4] Give the system of ordinary equations that results from using the expression in (a) and also give the initial and boundary conditions.

   (c) [3] Give the difference equations plus initial and boundary conditions that result from applying the forward Euler (or explicit Euler) to the system of ordinary differential equations.

   (d) [2] What is the advantage of applying the backward Euler (or implicit Euler) to the system of ordinary equations instead of forward Euler method?

Total [60]